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**Product differentiation in a mixed duopoly**

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**Abstract:** This paper analyzes a mixed duopoly with horizontal product differentiation using the unconstrained Hotelling model with quadratic transport costs. Firms play a noncooperative two-stage game on locations (first stage) and prices (second stage), and we consider that the firms move simultaneously or sequentially in both stages. We examine how the presence of a public firm affects both locations (the nature and degree of differentiation) and prices at equilibrium, and we compare these with the results of the private duopoly, in order to determine the effects of privatization. We find that, save in cases where the private firm is leader on prices where there are multiple equilibria, in the remainder of the cases there are two symmetrical equilibria (depending on whether one firm is located to the left or right of the other); the degree of differentiation of the product and the prices are lower; and the social welfare is higher than in the private duopoly. Privatization of the public firm is therefore not socially desirable.

*Keywords:* Mixed duopoly; spatial competition; privatization

*JEL classification:* L13, L32

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## **1. Introduction**

In many countries, it is common to observe both public and private firms competing within the same industry. Furthermore, the privatization process of public firms in numerous countries has given rise to analyses that have focused more on its economic justification than on ideological or political considerations. As a result, the study of mixed oligopolies has recently been paid considerable attention.

Industries such as airlines (India); television broadcasting (UK, Italy, Spain, France, Germany); the automotive industry (France); railways (Canada); banking (France, Portugal, Spain); oil (Norway, Spain); telecommunications (France, Spain); the postal service (Norway, Spain, and even the USA in package and overnight mail delivery); and natural gas (Spain) are examples of the fact that competition between private and public firms - mixed markets - has existed in the past or, indeed, still exists now.

Mixed markets have been very common in Western European countries since the 1960s. In the 1980s and 1990s the privatization program which started up in these countries spurred some markets to become private at national level, particularly in the UK, Austria, France, Italy, Portugal and Spain. In Spain, for instance, this occurred in a number of different markets: oil, where Repsol was privatized; banking, through Argentaria; the airlines, through Iberia, and electricity, through Endesa, etc...

In addition, a liberalization process developed which has propitiated the entry of private firms into sectors which traditionally public firms have tendered for on a monopoly basis (for example, postal services and telecommunications), thus giving rise to mixed markets. In turn, the process of economic integration amongst the EU countries, with the removal of borders and barriers to trade and firm location, has motivated the creation of new mixed markets where national firms compete with both public and private foreign firms, a consequence also of the differing degrees of privatization within EU countries. This has given rise to public companies in some countries (France, Italy, Germany) continuing to play a key role. For example, in Spain, with the privatization of Endesa, there are now no public Spanish firms in the Spanish electricity market. Nevertheless, this market is mixed due to the fact that the liberalization process triggered by the EU authorities led to public firms, Enel (Italy)

and EDP (Portugal), operating in that market. To sum up, mixed markets have played and still play a major role in the real economies of many countries.

The first works devoted to mixed economies considered a context of homogenous products (see Merrill and Schneider (1966), Harris and Wiens (1980), Beato and Mas-Colell (1984), Bös (1986, Ch. 11), Gil (1988), Cremer, Marchand and Thisse (1989) and De Fraja and Delbono (1989), (1990)). Subsequently, a series of contributions to the literature considered product differentiation, which also allowed for an analysis of the effects of privatization in this context. Cremer, Marchand and Thisse (1991), considered the Hotelling model (1929) with quadratic costs (see d'Aspremont, Gabszewicz and Thisse (1979)), where firms could only locate on the interval  $[0,1]$ , studied the optimum number and optimum locations of public firms that compete with private firms, determining under what conditions a mixed oligopoly is socially preferable to a private one. Using the monopolistic competition model, Anderson, de Palma and Thisse (1997) considered a market where one public firm competes with private firms and analyzed what happens when the public firm is privatized. Matsumura and Matsushima (2003), also taking the Hotelling model with quadratic costs, where firms are located on the interval  $[0,1]$ , investigated the sequential choice of location (the price stage is played simultaneously) in a mixed duopoly and studied the desirable role for the public firm (leader or follower). They also considered the effect of price regulation and determined that neither this nor the privatization of the public firm improved welfare.

Against this background, the present paper adopts the so-called unconstrained Hotelling model (see Lambertini (1994), (1997)), which is so named because firms can locate on any point of the real axis. Our aim is to analyze how the presence of a public firm - as opposed to a private duopoly - affects location and prices, as well as to determine the effects of that public firm becoming privatized. We also analyze how public firms acting as leader or follower affect prices and locations, considering that the private firm can also act as a leader or follower in both subgames.

In the subgame on prices, we find that the optimum strategy of the public firm, when it acts as follower, is to set a price equal to that of the private firm. This conditions the behavior of firms in the locations subgame, in particular that of the private firm, given that by reducing competition over prices, greater differentiation is not necessary, as occurs in the private duopoly.

When the two firms compete simultaneously in prices and locations, we obtain the locations corresponding to the social optimum, achieving the maximum possible social welfare. This result is maintained if the private firm acts as follower on prices and locations and the public firm is leader on prices and /or locations. Given that, in this context, when the public firm acts as follower in both subgames, the best social welfare is obtained. When the public firm acts as leader on price and/or location, it does not enable it to obtain more welfare. Therefore, the public firm role of leader or follower in both subgames is irrelevant. Our results show that product differentiation, which is strictly horizontal, is lower here than in the case of the private duopoly. This reduces prices. It is also shown that privatization is not socially desirable in this context.

In cases where the private firm acts as leader only in locations, the social optimum is not reached, since the private firm chooses the best location to achieve its objective of maximizing profit, which is the centre of the market. The public firm is located within the market, and product differentiation, which is strictly horizontal, is lower than in the private duopoly, which is weakly vertical, and with lower prices and greater social welfare. In this context privatization is not socially desirable. The fact that the public firm is leader or follower on prices does not affect results.

When the private firm is leader on prices, regardless of whether it is leader or follower on locations, we find ourselves with a particular problem which gives rise to multiple equilibria.

The rest of the paper is organized as follows. Section 2 presents the model and investigates the equilibrium outcomes. Section 3 concludes the paper.

## 2. Model and results

Consider Lambertini's duopoly model (1997) with a public firm. Firm 1 is a public firm which maximizes social welfare and firm 2 is a private firm which maximizes its own profits. The two firms supply the same physical product at different locations on the real axis. Production costs are nil. Consumers are uniformly distributed, with density one, along a linear city whose length can be normalized to 1 without loss of generality. They have unit demands, and consumption yields a positive constant surplus  $s$ ; each consumer buys if and only if the following condition is met:

$$U = s - td^2 - p_i \geq 0, \quad t > 0, \quad i = 1, 2; \quad (1)$$

where  $td^2$  is the transport cost incurred by a consumer living at distance  $d$  from store  $i$ , and  $p_i$  is the price of good  $i$ . It is assumed that the whole market is covered. Each consumer patronizes the firm giving the higher net surplus. Firm 1 is located at  $a$ , while firm 2 is located at  $1-b \geq a^1$ , with  $a, b \in \mathbb{R}^2$ , so that when both  $a$  and  $b$  are negative, firms are located outside the city boundaries. The demand functions are, respectively:

$$y_1 = a + \frac{1-a-b}{2} + \frac{p_2 - p_1}{2t(1-a-b)}, \quad y_1 \in [0,1]; \quad (2)$$

$$y_2 = 1 - y_1 = b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)}. \quad (3)$$

The private firm aims at maximizing its profits

$$\Pi_2 = p_2 y_2 = p_2 \left( b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \right) \quad (4)$$

For its part, the objective of the public firm is to maximize the social surplus. Given that individual demands are perfectly inelastic, this implies minimizing total transport costs. This is the case because, in these circumstances, prices above marginal cost do not give rise to distortions in the allocation of resources. The total social costs involved by transportation take the form:

$$SC = t \left[ \int_0^{y_1} (x - a)^2 dx + \int_{y_1}^1 (1 - b - x)^2 dx \right]. \quad (5)$$

Firms play a noncooperative two-stage game on locations (first stage) and prices (second stage). The solution concept is a subgame perfect equilibrium through backward induction.

As in Lambertini (1997), we refer to the nature of the differentiation at equilibrium considering the following definitions:

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<sup>1</sup> If  $1-b \leq a$ , we obtain symmetrical equilibria in prices and locations.

Definition 1. A noncooperative equilibrium in locations is strictly horizontal if, at equal prices, the indifferent consumer lies in (0,1).

Definition 2. A noncooperative equilibrium in locations is weakly vertical if, at equal prices, the indifferent consumer lies at 0 (or 1), while the others strictly prefer the right (left) firm. (Notice that an equilibrium which is weakly vertical is also weakly horizontal).

Definition 3. A noncooperative equilibrium in locations is strictly vertical if, at equal prices, one firm is strictly preferred by all consumers.

## 2.1. Simultaneous moves

Both firms compete simultaneously in prices and locations.

*Proposition 1:* When firms compete simultaneously in prices and locations, the Nash equilibrium in locations is strictly horizontal and the locations are determined by  $a=b=1/4$  ( $1-b=3/4$ ). Privatization of the public firm does not improve welfare.

*Proof:* Minimizing (5) with respect to  $p_1$ , we obtain:

$$(p_2 - p_1) \frac{\partial y_1}{\partial p_1} = 0.$$

As  $\partial y_1 / \partial p_1 \neq 0$  by (2), public firm 1 sets its price at the same level as firm 2, i.e.,  $p_1 = p_2$ . (6)

Maximizing (4) with respect to  $p_2$ , and considering (6), we obtain the equilibrium prices:

$$p^*_1(a, b) = p^*_2(a, b) = t(1 - a - b)(1 - a + b) \quad (7)$$

that differ from those obtained in the private duopoly case (see Lambertini, 1994).

Substituting the price equilibrium in (4) and (5), we obtain

$$SC = t \left[ \frac{(1 - a - b)^3}{12} + \frac{a^3}{3} + \frac{b^3}{3} \right] \quad (8)$$

$$\Pi_2 = \frac{t}{2} (1 - a - b)(1 - a + b)^2 \quad (9)$$

Differentiating (8) and (9) with respect to  $a$  and  $b$  respectively, we obtain  $a^*=b^*=1/4$  and  $1-b^*=3/4$ . The consideration of a mixed duopoly allows us to determine the best locational configuration.<sup>2</sup> The equilibrium profits are  $\Pi_1=\Pi_2=t/4$ ,  $SC=t/48$ , demands are  $y_1=y_2=1/2$  and prices are  $p_1=p_2=t/2$ .

This result differs from that obtained by Lambertini (1994) for a private duopoly:  $a^*=b^*=-1/4$ , which implied that the firms locate symmetrically outside the city, and that the equilibrium profits are  $\Pi_1=\Pi_2=3t/4$ ,  $SC=13t/48$ , demands  $y_1=y_2=1/2$  and prices  $p_1=p_2=3t/2$ .

Therefore, it would appear that we have a strong argument in favor of a public firm. There is a move from a very inefficient locational configuration ( $a=b=-1/4$ ) to an efficient one ( $a=b=1/4$ ,  $1-b=3/4$ ). The SC is smaller in a mixed duopoly than in a private one. The mixed duopoly is socially preferable to its private counterpart.

Finally, both the firms' profits and prices are lower in the case of the mixed duopoly.

## 2.2. Sequential moves

### 2.2.1. Location leadership

In this case, the price stage is played simultaneously and the location stage is played sequentially.

Consider two situations:

- A) The public firm acts as a Stackelberg leader at the location stage.
- B) The private firm acts as a Stackelberg leader at the location stage.

*Proposition 2:* When firms compete simultaneously in prices, if the public firm acts as a Stackelberg leader at the location stage, the locations are  $a=b=1/4$  ( $1-b=3/4$ ) and if the private firm acts as a Stackelberg leader at the location stage, the locations are  $a=1/6$ ,  $b=1/2$  ( $1-b=1/2$ ). In neither case is privatization socially desirable.

*Proof:* On solving the subgame in prices, we obtain (7), which we substitute in the objective functions of the public and private firms, obtaining (8) and (9).

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<sup>2</sup> This result coincides with that obtained by Cremer, Marchand and Thisse (1991) who considered the locations of the firms solely on the interval  $[0,1]$ .

In the case of A), the public firm is leader in locations. The objective of the private firm at the location stage is:

$$\max_b \Pi_2 = \frac{t}{2}(1-a-b)(1-a+b)^2 \quad (10)$$

Considering the first and second order conditions of (10), we obtain

$$b = \frac{1-a}{3}. \quad (11)$$

The objective of the public firm at the location stage is:

$$\min_a SC = t \left[ \frac{(1-a-b)^3}{12} + \frac{a^3}{3} + \frac{b^3}{3} \right] \quad (12)$$

$$\text{s.t. } b = \frac{1-a}{3}.$$

Consider the first and second order conditions of problem (12), we obtain  $a=b=1/4$ ,  $1-b=3/4$ . The equilibrium demands are  $y_1=y_2=1/2$ ; prices are  $p_1=p_2=t/2$ ; profits are  $\Pi_1=\Pi_2=t/4$  and  $SC=t/48$ .

These results differ from those obtained by Lambertini (1994, 1997) for the private duopoly, which, considering that firm 1 acts as leader, implies:  $a = 1/2$ ,  $b = -1/2$ ,  $1-b = 3/2$ , so that the leader firm is located in the centre of the market and the other is outside the city (if firm 2 acts as leader, the result is symmetrical:  $a = -1/2$ ,  $b = 1/2$ ,  $1-b = 1/2$ ). The demands are:  $y_1 = 2/3$ ,  $y_2 = 1/3$ ; prices are:  $p_1 = 4t/3$ ,  $p_2 = 2t/3$ ; profits are:  $\Pi_1 = 8t/9$ ,  $\Pi_2 = 2t/9$  and  $SC = 7t/36$ .

Once again, it can be seen that there is a strong argument in favor of the public firm, given that there is a transition from a configuration of inefficient locations in the private duopoly ( $a = 1/2$ ,  $1-b = 3/2$ ) to an efficient location in a mixed duopoly ( $a = 1/4$ ,  $1-b = 3/4$ ). We obtain the same result as in the case where the public firm is a follower in locations: since the maximum social welfare possible was obtained, the fact that the public firm has an advantage in the subgame in locations does not allow an increase in the level of welfare. Therefore, the fact that the public firm is leader or follower in this context is irrelevant. Observe that the degree of product differentiation in the mixed duopoly is lower, as are the prices.



In case B) the private firm is leader in location. The objective of the public firm in the location stage is:

$$\min_a SC = t \left[ \frac{(1-a-b)^3}{12} + \frac{a^3}{3} + \frac{b^3}{3} \right] \quad (13)$$

Considering the first and second order conditions of (13), we obtain

$$a = \frac{1-b}{3}. \quad (14)$$

The objective of the private firm at the location stage is:

$$\max_b \Pi_2 = \frac{t}{2} (1-a-b)(1-a+b)^2 \quad (15)$$

$$\text{s.t. } a = \frac{1-b}{3}$$

Considering the first and second order conditions of (15), we obtain  $a=1/6$ ,  $b=1/2$ ,  $1-b=1/2$ .

The equilibrium demands are  $y_1=1/3$ ,  $y_2=2/3$ ; prices are  $p_1=p_2=4t/9$ ; profits are  $\Pi_1=4t/27$ ;  $\Pi_2=8t/27$  and  $SC=5t/108$ .

If we compare this with the results obtained by Lambertini (1997), we observe that the location of the leader firm is the same in both cases. This is not the case of the follower firm which, in the mixed duopoly, is the public firm, being located within the market, at  $1/3$  of the distance from the leader. In the private duopoly case, however, it is located outside the market at one unit of distance from the leader. The consideration of a public firm implies that we pass from private duopoly locations:  $a=1/2$ ,  $1-b=1/2$  to mixed duopoly locations:  $a=1/6$ ,  $1-b=1/2$ , so that the degree of differentiation is less in the mixed duopoly. However, the social optimum is not reached, given that the private firm which is leader in locations chooses the most advantageous location, namely the centre of the market.

From the comparison of the results of A) and B) <sup>3</sup> we observe that in the mixed duopoly case, social welfare is greater when the public firm acts as leader than when the private firm does.

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<sup>3</sup> The results obtained in both cases agree with those obtained by Matsumura and Matsushima (2003) who consider that firms can only locate at  $[0,1]$ .

In both cases, privatization of the public firm would give rise to a loss of social welfare, with this loss being greater when the public firm is leader in location. Privatization is, therefore, not socially desirable. In both cases, the differentiation is strictly horizontal, as opposed to the weakly vertical differentiation of the private duopoly.

### 2.2.2. Price leadership.

The price stage is played sequentially, while the location stage is played simultaneously.

Consider two situations:

- C) The public firm acts as a Stackelberg leader in the price stage.
- D) The private firm acts as a Stackelberg leader in the price stage.

For case C) we obtain,

*Proposition 3:* When firms play simultaneously in locations, if the public firm acts as a Stackelberg leader in the price stage, the locations are  $a=b=1/4$  ( $1-b=3/4$ ), obtaining the social optimum. Privatization of the public company does not improve welfare.

*Proof:* When the public firm acts as a Stackelberg leader in the price stage, the objective of the private firm in the price stage is:

$$\max_{p_2} \Pi_2 = p_2 \left( b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \right) \quad (16)$$

Considering the first order conditions of (16), we obtain

$$p_2 = \frac{p_1}{2} + \frac{t}{2} (1-a-b)(1-a+b). \quad (17)$$

The objective of the public firm in the price stage is:

$$\min_{p_1} SC = t \left[ \int_0^{y_1} (x-a)^2 dx + \int_{y_1}^1 (1-b-x)^2 dx \right] \quad (18)$$

$$\text{s.t. } p_2 = \frac{p_1}{2} + \frac{t}{2} (1-a-b)(1-a+b).$$

From the first and second order conditions of (18), we obtain (7).

We obtain the same prices as in the case of simultaneous moves, and given that in the locations subgame the firms play simultaneously, we obtain the same result as in the case of simultaneous moves:  $a=b=1/4$ ,  $1-b=3/4$ . The equilibrium demands are  $y_1=y_2=1/2$ ; prices are  $p_1=p_2=t/2$ ; profits are  $\Pi_1=\Pi_2=t/4$  and  $SC=t/48$ .

These results differ from those obtained by Lambertini (1997) for the private duopoly, considering that firm 1 acts as leader:  $a=0$ ,  $b=-1$ ,  $1-b=2$ . Once again, there is a strong argument in favor of the public firm, given that we move from an inefficient configuration of locations in the private duopoly ( $a=0$ ,  $1-b=2$ ) to an efficient one ( $a=1/4$ ,  $1-b=3/4$ ).

The differentiation in the mixed duopoly is strictly horizontal, as opposed to the weakly vertical differentiation in the private duopoly. The prices are lower in the mixed duopoly. When comparing this with the case where firms move simultaneously in the two subgames, we observe that we obtain the same result. Furthermore, given that the maximum social welfare had already been obtained there, the fact that the public firm has an advantage in the prices subgame does not allow it to obtain more welfare. Therefore, if both firms move simultaneously in locations and the private firm acts as a follower in prices, it is irrelevant whether the public firm acts as leader or follower in prices. The role of leader gives no advantage to the public firm. This contrasts with the result obtained by Lambertini (1997, see Corollary 3.1), where private firms prefer to act as leaders in prices.

For case D) we have a particular problem with multiple equilibria, depending on the values of  $s$  and  $t$ .

*Proposition 4:* When the private firm acts as a Stackelberg leader in the price stage, while firms move simultaneously in the location stage, the prices are  $p_1=p_2=s-tb^2$

$$\text{and the locations are } a = \frac{13 - \sqrt{4 + 33\frac{s}{t}}}{33}, \quad b = \frac{-2 + \sqrt{4 + 33\frac{s}{t}}}{11}. \quad (1-$$

$$b = \frac{13 - \sqrt{4 + 33\frac{s}{t}}}{11}). \text{ If } \frac{s}{t} = \frac{9}{16} \text{ then } a=b=1/4, \quad 1-b=3/4.$$

*Proof:* When the private firm acts as a Stackelberg leader in the price stage, the objective of the public firm in the price stage is:

$$\min_{p_1} SC = t \left[ \int_0^{y_1} (x - a)^2 dx + \int_{y_1}^1 (1 - b - x)^2 dx \right] . \quad (19)$$

From the first order conditions of (19), we obtain  $p_1 = p_2$ .

The objective of the private firm in the price stage is:

$$\max_{p_2} \Pi_2 = p_2 \left( b + \frac{1-a-b}{2} + \frac{p_1 - p_2}{2t(1-a-b)} \right) \quad (20)$$

$$\text{s.t. } p_1 = p_2.$$

Substituting in (20), we obtain the following problem:

$$\max_{p_2} \Pi_2 = p_2 \left( b + \frac{1-a-b}{2} \right) \quad (21)$$

where, as the profit of the private firm is linear at  $p_2$ , given the locations, we obtain the result that the maximum profit is reached for the highest value that  $p_2$  can take. This will depend on  $s$  and  $t$ , assuming that the whole market is covered. Specifically, the maximum value that  $p_2$  can take, given the locations and covering the whole market, allows two possibilities:

i)  $s = p_2 + t(1 - (1-b))^2 = p_2 + tb^2$  that implies  $p_2 = s - tb^2$ , with  $p_1 + t(y_1 - a)^2 = p_2 + t((1-b) - y_1)^2 \leq s$  and  $p_1 + ta^2 \leq s$ .

ii)  $p_1 + t(y_1 - a)^2 = p_2 + t((1-b) - y_1)^2 = s$  that implies  $p_2 = s - \frac{t(1-a-b)^2}{4}$ , with  $p_1 + ta^2 \leq s$  and  $p_2 + tb^2 < s$ .

In case i) with  $p_2 = s - tb^2$ , we have the problem:

$$\max_b \Pi_2 = (s - tb^2) \left( b + \frac{1-a-b}{2} \right) \quad (22)$$

and from the first and second order conditions of (22), we obtain

$$b = \frac{-(1-a) + \sqrt{(1-a)^2 + 3\frac{s}{t}}}{3} . \quad (23)$$

The problem of the public firm is (13), whose solution is (14).

From (23) and (14), considering the second order conditions, we obtain

$$a = \frac{13 - \sqrt{4 + 33\frac{s}{t}}}{33}, b = \frac{-2 + \sqrt{4 + 33\frac{s}{t}}}{11}. (1-b = \frac{13 - \sqrt{4 + 33\frac{s}{t}}}{11}).$$

If  $\frac{s}{t} = \frac{9}{16}$  then  $a=b=1/4$ ,  $1-b=3/4$ . The equilibrium demands are  $y_1=y_2=1/2$ ; prices are  $p_1=p_2=t/2$ ; profits are  $\Pi_1=\Pi_2=t/4$ ; and  $SC=t/48$ .

Given that  $+\sqrt{4 + 33\frac{s}{t}}$  is greater than 2, we have  $b>0$ , which implies that  $1-b<1$ .

Consequently, considering that  $1/2 \leq 1-b$ , due to the risk of leapfrogging by the other firm, we have  $1/2 \leq 1-b<1$ , and considering that  $a=\frac{1-b}{3}$  we have  $1/6 \leq a<1/3$ .

Consequently, the differentiation will be strictly horizontal and less than in the private duopoly case where, for the case of firm 2 leader in prices, we have  $a = -1$ ,  $b=0$  and  $1-b=1$ , with weakly vertical differentiation. This would imply greater social welfare and lower prices in the mixed duopoly.

In case ii) there is no solution. Now we have  $p_2=s-\frac{t(1-a-b)^2}{4}$  and the problem

$$\max_b \Pi_2 = (s - \frac{t(1-a-b)^2}{4})(b + \frac{1-a-b}{2}) \quad (24)$$

From the conditions of first and second order from (24), we obtain:

$$b = \frac{1-a}{3} + \frac{2\sqrt{(1-a)^2 + 3\frac{s}{t}}}{3} \quad (25)$$

The problem of the public firm is (13), whose solution is (14).

From (25) and (14), considering the second order conditions, we obtain:

$$a = \frac{1 - \sqrt{1 + 5\frac{s}{t}}}{5}. \quad (26)$$

Considering (14) and the second order condition of (13):  $a > -\frac{(1-b)}{3}$ , we obtain that  $a > 0$ , (26) does not hold and, therefore, there is no solution.

This case occurs when firm 2 is located to the right of  $3/4$ . However, as the public firm sets the same price as the private firm, and the market must be covered, the firms do not have the incentive to separate as much as possible from each other, which would enable them to raise the prices in the private duopoly. Now, the more they distance themselves from each other, with the same prices and with the restriction of covering the market, the lower the price: this prevents any possible solution. It is not in firm 2's interest to be located to the right of  $3/4$ .

### 2.2.3. Alternate leadership.

In each subgame, one of the firms acts as leader and the other as follower, taking turns at leadership.

Consider two situations:

- E) The public firm acts as a Stackelberg leader in the price stage and the private firm does so in the location stage.
- F) The private firm acts as a Stackelberg leader in the price stage and the public firm does so in the location stage.

First, we consider case E).

*Proposition 5:* If the private firm acts as Stackelberg leader in the location stage, while the public firm acts as leader in the price stage, then the public firm locates at  $a=1/6$  and the private firm locates in the centre of the city,  $1-b=1/2$ , so that the Nash equilibrium in locations is horizontally differentiated. Privatization is not socially desirable.

*Proof:* From the subgame on prices, where the public firm acts as a Stackelberg leader and the private firm as a follower, we obtain (7).

Consider (7), and substituting in (4) and (5), we consider the subgame on locations where the private firm acts as a Stackelberg leader in locations. The objective of the public firm in the location stage is (13) and we obtain (14). The objective of the

private firm in the location stage is (15) and we obtain the same results as in case B):  $a=1/6$ ,  $b=1/2$ ,  $1-b=1/2$ . Equilibrium demands are  $y_1=1/3$ ,  $y_2=2/3$ ; prices are  $p_1=p_2=4t/9$ ; profits are  $\Pi_1=4t/27$ ,  $\Pi_2=8t/27$  and  $SC=5t/108$ .

The same result is obtained as in the case where the private firm acted as a Stackelberg leader in location, and both firms acted simultaneously in prices (case B). Therefore, if the private firm acts as a leader in locations and as a follower in prices, the role of the public firm (leader or follower) is irrelevant in the price subgame.

If we compare this with Lambertini's results (1997) for the private duopoly, considering that firm 1 acts as leader in prices and firm 2 as leader in locations:  $a = -1/3$ ,  $b = 0$ ,  $1-b = 1$ ,  $y_1=1/3$ ,  $y_2=2/3$ ,  $p_1=p_2=16t/9$ , a greater degree of differentiation is observed in the private duopoly, This causes higher prices, specifically four times higher, and a lower level of social welfare. In both the private and mixed duopoly, differentiation is strictly horizontal. Privatization of the public firm is not socially desirable.

For case F), we obtain:

*Proposition 6:* When the private firm acts as a Stackelberg leader in the price stage, while the public firm acts as a Stackelberg leader in the location stage, there are multiple equilibria, depending on the values of  $s$  and  $t$ . Specifically, if  $\frac{s}{t} = \frac{9}{64}$  then  $a=b=1/4$ ,  $1-b=3/4$ .

*Proof:* The price stage is described by (19-21 and i), and the location stage by the private firm is described by (22-23). The objective of the public firm in the location stage is then:

$$\min_a SC = t \left[ \frac{(1-a-b)^3}{12} + \frac{a^3}{3} + \frac{b^3}{3} \right] \quad (27)$$

$$\text{s.t. } b = \frac{-(1-a) + \sqrt{(1-a)^2 + 3\frac{s}{t}}}{3}.$$

From the first order conditions of (27), we obtain:

$$31(1-a)^6 + 144a^4(1-a)^2 - 160a^2(1-a)^4 + 82r(1-a)^4 + 432ra^4 - 480ra^2(1-a)^2 - 49r^2(1-a)^2 = 0, \quad (28)$$

with  $r = s/t$ .

It does not appear to be easy to find an analytical solution to this equation.

Giving values to  $r$  in (28), we obtain different equilibria. In particular if  $r = \frac{s}{t} = \frac{9}{16}$  then  $a=b=1/4$ ,  $1-b=3/4$ . The equilibrium demands are  $y_1=y_2=1/2$ , prices are  $p_1=p_2=t/2$ , profits are  $\Pi_1=\Pi_2=t/4$  and  $SC=t/48$ .

Given that  $+\sqrt{(1-a)^2 + 3\frac{s}{t}}$  is greater than  $1-a$ , we have it that  $b>0$ , which implies that  $1-b<1$ . Therefore, considering that  $1/2 \leq 1-b$ , due to the risk of leapfrogging by the other firm, we have  $1/2 \leq 1-b<1$ .

#### 2.2.4. Repeated leadership.

In each subgame, the same company acts as leader and the other as follower.

Consider two situations:

G) The public firm acts as a Stackelberg leader in both prices and locations.

H) The private firm acts as a Stackelberg leader in both prices and locations.

First, we consider the case where the Stackelberg leadership is assigned to the public firm in both stages.

*Proposition 7:* If the public firm acts as Stackelberg leader in both prices and locations, the locations are  $a=b=1/4$  ( $1-b=3/4$ ). Privatization is not socially desirable.

*Proof:* The price stage is described by Equations (16-18), obtaining (7). The objective of the public firm in the location stage is (12) and we obtain the same results as in case A):  $a=b=1/4$ ,  $1-b=3/4$ . Equilibrium demands are  $y_1=y_2=1/2$ ; prices are  $p_1=p_2=t/2$ ; profits are  $\Pi_1=\Pi_2=t/4$  and  $SC=t/48$ .

When Stackelberg leadership is assigned to the public firm in both stages, we obtain the same result as in the case where the two firms move simultaneously in the two subgames. Once again, the advantage of being leader in prices and locations does



not make it possible to obtain more social welfare. Therefore, if the private firm acts as follower in both subgames, the role of the public firm (leader or follower) in both subgames is irrelevant.

If we compare the results obtained by Lambertini (1997), for the case of firm 1 leader in prices and locations:  $a = 1/2$ ,  $b = -7/6$ ,  $1-b = 13/6$ , once again we observe a greater degree of differentiation in the private duopoly. This implies higher prices and a lower level of social welfare compared to the mixed duopoly. Privatization of the public firm is not socially desirable. In the private duopoly, differentiation is strictly vertical, whereas in the mixed duopoly it is strictly horizontal.

For case H), we obtain:

*Proposition 8:* If the private firm acts as Stackelberg leader in both prices and

locations, the prices are  $p_1=p_2=s-tb^2$  and the locations are  $a = \frac{7 - \sqrt{1 + 12\frac{s}{t}}}{18}$ ,  $b = \frac{-1 + \sqrt{1 + 12\frac{s}{t}}}{6}$ .  $(1-b = \frac{7 - \sqrt{1 + 12\frac{s}{t}}}{6})$ . If  $\frac{s}{t} = \frac{7}{16}$  then  $a=b=1/4$ ,  $1-b=3/4$ .

*Proof:* The price stage is described by (19-21 and i) and the location stage by the public firm is described by (13-14). The leader's problem in the location stage is thus:

$$\max_b \Pi_2 = (s - tb^2)(b + \frac{1-a-b}{2}) \quad (29)$$

$$\text{s.t. } a = \frac{1-b}{3}.$$

From the first and second order conditions of (29), we obtain

$$a = \frac{7 - \sqrt{1 + 12\frac{s}{t}}}{18}, b = \frac{-1 + \sqrt{1 + 12\frac{s}{t}}}{6}. (1-b = \frac{7 - \sqrt{1 + 12\frac{s}{t}}}{6}).$$

If  $\frac{s}{t} = \frac{7}{16}$  then  $a=b=1/4$ ,  $1-b=3/4$ . The equilibrium demands are  $y_1=y_2=1/2$ ,

prices are  $p_1=p_2=3t/8$ , profits are  $\Pi_1=\Pi_2=3t/16$  and  $SC=t/48$ .

Given that  $1 + \sqrt{1 + 12 \frac{s}{t}}$  is greater than 1, we have that  $b > 0$ , which implies that  $1 - b < 1$ . Therefore, considering that  $1/2 \leq 1 - b$ , due to the risk of leapfrogging by the other firm, we have  $1/2 \leq 1 - b < 1$ , and considering that  $a = \frac{1 - b}{3}$ , we have that  $1/6 \leq a < 1/3$ . Consequently, differentiation will be strictly horizontal and less than in the case of the private duopoly, which is strictly vertical. This causes lower prices and a higher level of social welfare in the mixed duopoly.

### 3. Conclusions

This paper has considered the unconstrained Hotelling model with a public firm, whose objective is social surplus maximization. It has also been assumed that both the public and private firm may act as leaders or followers in price and location. The results obtained have been compared with those obtained by Lambertini (1994, 1997), which consider only private firms.

The results obtained show that, in the price subgame, the best strategy of the public firm, when it acts as follower, is to set a price equal to that of the private firm. Even when the public firm acts as leader in prices, the same price for the two firms is obtained. This result differs from that found in the private duopoly case where, except in cases of simultaneous moves and alternative leadership, the firms set different prices, with the firm that is leader in prices and/or locations, setting a higher price. The fact that the competition in prices is less in the mixed duopoly has an effect on firm locations and on the type of product differentiation. Intuition seems to indicate that the public firm, in order to achieve maximum social welfare, mitigates competition in prices, so that major product differentiation is not necessary i.e. firms do not locate very far from each other.

In the locations subgame, except in the case of alternative leadership with the private firm leader in prices, we find that the public firm, when it acts as leader or follower in location, locates itself at a distance that is a third of that of the private firm i.e., relatively close by. This means that both firms are always located within the market, a fact which contrasts with the results of the private duopoly.

In the mixed duopoly, except when the private firm is leader in prices and/or locations, in all other cases locations corresponding to the social optimum are obtained. This gives the maximum possible social welfare and the degree of product differentiation and prices are lower than in the private duopoly. Privatization of the public firm is not socially desirable. Moreover, in these cases the role of leader or follower in prices and/or locations for the public firm is irrelevant.

When the private firm acts as leader in locations, without considering the case where it is also leader in prices, our results show that in the mixed duopoly the private firm is located in the centre of the market, as is the firm which is leader in locations in the private duopoly. However, the public firm is located at a distance of one third, which is less than the distance corresponding to the private duopoly. Consequently, the degree of product differentiation is less and prices are lower, while social welfare is greater than in the private duopoly. Once again, privatization of the public firm is not socially desirable and it is irrelevant whether the public firm is leader or follower in prices.

In cases where the private firm is leader in prices, we have a particular problem that involves us having multiple equilibria, depending on the values of  $s$  and  $t$ .

In the mixed duopoly, differentiation is strictly horizontal. This contrasts with the results of the private duopoly, where differentiation may be strictly horizontal, weakly vertical, and strictly vertical. We can equally conclude that comparison with the private duopoly confirms that the public firm is a good regulating instrument.

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